

Free Differential Algebras, Rheonomy and Pure Spinors

Pietro Fré¹ and Pietro Antonio Grassi²

¹ Dipartimento di Fisica Teorica, Università di Torino,
& INFN - Sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy

² DISTA, Università del Piemonte Orientale,
Via Bellini 25/G, Alessandria, 15100, Italy & INFN - Sezione di Torino

Abstract

We report on progresses on the derivation of pure spinor constraints, BRST algebra and BRST invariant sigma models a la pure spinors from the algebraic structure of the FDA underlying supergravity.

Talk given by Pietro Fré at the Workshop Supersymmetry and Quantum Symmetry 2007 held at the Joint Institute for Nuclear Research in Dubna (Russian Federation), July 2007

1 Introduction

A fundamental problem ubiquitous in current research on string theory is that of calculating string amplitudes in the presence of background Ramond Ramond fields. Neither the Neveu Schwarz nor the Green Schwarz formulation of the string sigma model is apt to such a task because these formulations treat the Ramond-Ramond background fields either non-polynomially (by means of spin fields) or non-covariantly (using light-cone gauge). An additional problem in the case of the Green Schwarz formulation is located in the BRST quantization of its distinctive local symmetry namely κ -supersymmetry. The non-conventional nature of this symmetry which classically is nothing else but the pull-back on the world-sheet of half of the supersymmetry transformations of bulk supergravity, gives origin to an infinite hierarchy of ghosts for ghosts.

An apparently thorough solution of all these problems has been provided by the Berkovits' pure spinor reformulation of the string sigma model [1]. Berkovits' construction is an extended version of the Green Schwarz formulation. In addition to the classical fields it introduces a triplet of new fields, λ^i, w_i, d_i which are all target space 32-components spinors. The last two members of the triplet w_i, d_i are also world-volume vectors as denoted by the index i . This triplet is the classical one of BRST quantization, the three members being respectively characterized by ghost number 1, $-1, 0$ and admitting therefore the interpretation of ghost, antighost and Lagrange multiplier. Indeed in Berkovits approach the starting point is provided by the definition of a BRST operator :

$$Q^{BRST} = \int \lambda^i d_i d^2 \sigma^i \quad (1.1)$$

against which the string sigma-model is required to be invariant. At the same time the BRST charges are requested to be nilpotent. Although only in a sense yet to be clarified, it is clear of which symmetry the λ -fields, which are commuting, are supposed to be the ghosts: this is bulk supersymmetry pull-backed onto the world sheet. Just as κ -supersymmetry. The catch of the method is provided by the constraints of the type:

$$\bar{\lambda} \Gamma^a \lambda \approx 0 \quad (1.2)$$

which the superghosts λ are requested to satisfy. In Berkovits approach the constraints of type (1.2), named pure spinor constraints are an a-priori input which constitutes part of the definition of the BRST operator. At a second stage of development, in Berkovits approach, one tries to determine the constraints on target superspace geometry required for BRST invariance of the sigma model action. For full consistency of the approach it should happen that these latter constraints be those describing target space supergravity. Although this can be achieved through elaborate steps [9], yet in this approach a clearcut correspondence between the background bulk geometry and the pure spinor sigma model is not available. This makes it difficult to perform an immediate direct construction of the pure spinor sigma model for any chosen supergravity background. This becomes particularly evident when one considers backgrounds with a reduced number of conserved supersymmetries like $\text{AdS}_5 \times \text{T}^{1,1}$ for the type IIB case, the compactification on $\text{AdS}_3 \times \text{CP}^3$ for the type IIA theory.

It must also be noted that the pure spinor constraints (1.2) are of the general type indicated but, depending on the chosen theory (type IIA, type IIB or M-theory) and on the chosen background, have to be tuned in their explicit form.

For this reasons it would be highly desirable to have a formulation of the pure spinor sigma models in which the pure spinor constraints, the BRST operator and the entire set up follow from background supergravity just as it happens for the κ -supersymmetric actions.

Such a formulation is in progress. Work has been done in [11, 14] in the case of the M2-brane and new results are upcoming for the case of type II superstrings [13, 12].

The general idea is encoded in the following list of constructive steps:

Flow chart to construct a SUGRA

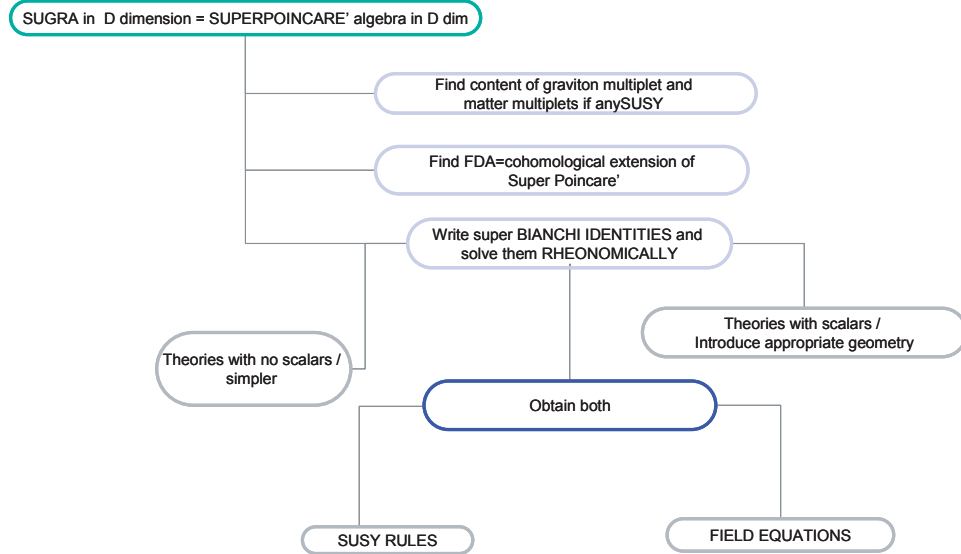


Table 1: *This figure illustrates the flow chart for the construction of supergravity theories. From the superalgebra to the field equations and supersymmetry transformation rules everything is encoded in the structure constants of the superalgebra plus the principle of rheonomy*

1. The algebraic structure underlying any higher dimensional supergravity theory is a Free Differential Algebra (FDA). This latter is a categorical extension of a (super) Lie

algebra determined by the Chevalley cohomology of this latter.

2. Given the FDA one considers its Bianchi identities and constructs the unique rheonomic parametrization of the FDA curvatures. Rheonomy is a universal principle of analiticity in superspace (see fig.2) which requires that the fermionic components of the FDA curvatures should be linear functions of their bosonic ones. Rheonomy encodes in one single principle the construction of both field equations and supersymmetry transformation rules for any supergravity. Indeed field equations follow as integrability conditions of the rheonomic parametrization of curvatures. The flow chart for the construction of classical supergravities and the principle of rheonomy are respectively illustrated in table 1 and 2
3. Consider then the FDA appropriate to the supergravity under investigation and the rheonomic parametrization of its curvatures.
4. Perform the ghost-form extension of the classical FDA according to the principle introduced by Anselmi and Fré in [8], namely:

Principle 1.1

The correct BRST algebra is provided by replacing, in the rheonomic parametrization, of the classical supergravity curvatures each differential form with its extended ghost-form counterpart while keeping the curvature components untouched. Thus one obtains the rheonomic parametrization of the ghost-extended curvatures, whose formal definition is identical with that of the classical curvatures upon the replacements:

$$\begin{aligned} d &\mapsto d + \mathcal{S} \\ \Omega^{[n]} &\mapsto \sum_{p=0}^n \Omega^{[n-p,p]} \end{aligned} \tag{1.3}$$

In this way one has the ordinary (unconstrained) BRST algebra of supergravity.

5. Set to zero all the bosonic ghosts. This defines a constrained BRST algebra and for consistency a certain set of **pure spinor** constraints. The correct constraints are the projection onto the world-sheet (brane world volume) of these constraints. The pure spinor constraints should not be chosen a priori as an input.¹
6. Verify that the pure spinor constraints can be solved in terms of as many independent degrees of freedom as it is required for a $c = 0$ conformal theory in $d=2$ in the case of superstrings.
7. Introduce the appropriate antighosts and Lagrange multiplier field and construct the BRST invariant quantum action.

¹We point out that the relation between pure spinor formulation and extended supersymmetry algebras [3] has been discovered in [4]. Recently, also N. Berkovits pointed out the relation between the rheonomic parametrization and the pure spinor superspace [5].

Rheonomy of superspace

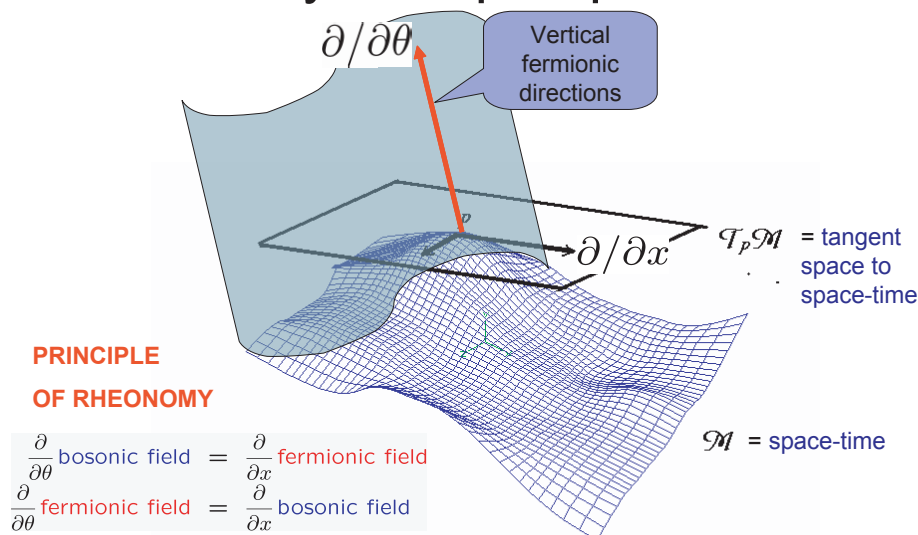


Table 2: *The principle of rheonomy is an analogue of the Cauchy Riemann equations for analytic functions. Just as the derivative of the imaginary part v of $f(z)$ in the x direction is related to the derivative of the real part u in the y direction, in the same way the fermionic derivative of bosonic fields is expressed as a combination of the bosonic derivatives of the fermionic ones. This is summarized by requiring that all external components (fermionic) of the FDA curvature should be given as linear functions of the inner components of the FDA curvatures. There is also an analogue of the differential equation satisfied by u and v . This analogue are the field equations of supergravity which follow as integrability conditions of the rheonomic parametrization of FDA curvatures*

In this talk we review the application of this scheme to the case of M-theory (having in mind the M2-brane which was discussed in our common paper [11]). We use this example to illustrate the flow chart of the construction. In particular we focus on points 1-6 of the above list. Our main goal is to show how the structure of the BRST algebra for all the fields with non negative ghost number together with the pure spinor constraints are completely determined by the original superPoincaré algebra through the following algorithmic steps which at each level yield the unique result displayed in table 3: The discussion of the antighost sector and of the BRST invariant action is not treated here. It is the subject of the forthcoming papers [13, 12]

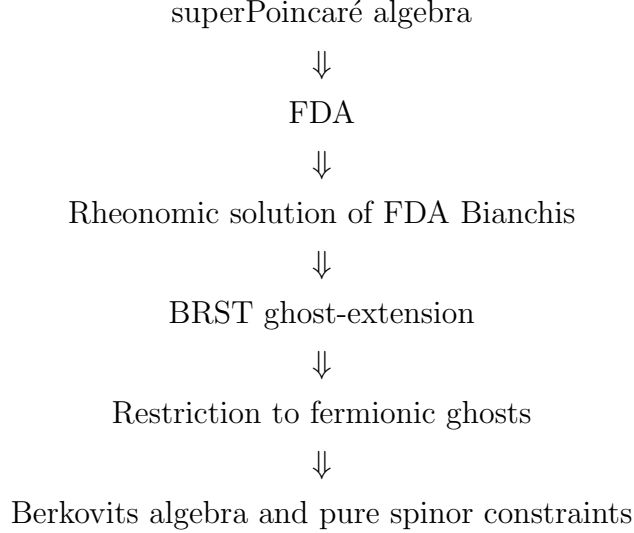


Table 3: The deterministic path from the super Poincaré algebra to the constrained BRST algebra with pure spinors

2 General Structure of FDA.s and Sullivan's theorems

Free Differential Algebras (FDA) are a natural categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (p -forms) of rank greater than one.

FDA.s were independently discovered in Mathematics by Sullivan [2] and in Physics by the present author in collaboration with R. D'Auria [3]. The original name given to this algebraic structure by D'Auria and me was that of *Cartan Integrable Systems*. Later, recognizing the conceptual identity of our supersymmetric construction with the pure bosonic constructions considered by Sullivan, we also turned to its naming FDA which has by now become generally accepted.

Let me recall the definition of FDA.s and two structural theorems by Sullivan which show how all possible FDA.s are, in a sense to be described, *cohomological extensions* of normal Lie algebras or superalgebras.

Another question which is of utmost relevance in all physical applications is that of *gauging* of FDA.s. Just in the same way as physics gauges standard Lie algebras by means of Yang Mills theory through the notion of gauge connections and curvatures one expects to gauge FDA.s by introducing their curvatures. A surprising feature of the FDA setup which was noticed and explained by me in a paper of 1985 [6] is that differently from Lie algebras the algebraic structure of FDA already encompasses both the notion of connection and the notion of curvature and there is a well defined mathematical way of separating the two which

relies on the two structural theorems by Sullivan.

Definition of FDA The starting point for FDA.s is the generalization of Maurer Cartan equations. A standard Lie algebra is defined by its structure constants which can be alternatively introduced either through the commutators of the generators:

$$[T_I, T_K] = \tau^I_{JK} T_I \quad (2.1)$$

or through the Maurer Cartan equations obeyed by the dual 1-forms:

$$de^I = \frac{1}{2} \tau^I_{JK} e^J \wedge e^K \quad (2.2)$$

The relation between the two descriptions is provided by the duality relation:

$$e^I(T_J) = \delta^I_J \quad (2.3)$$

Adopting the Maurer Cartan viewpoint FDA.s can now be defined as follows. Consider a formal set of exterior forms $\{\theta^{A(p)}\}$ labelled by the index A and by the degree p which may be different for different values of A . Given this set we can write a set of generalized *Maurer Cartan equations* of the following type:

$$d\theta^{A(p)} + \sum_{n=1}^N C^{A(p)}_{B_1(p_1)\dots B_n(p_n)} \theta^{B_1(p_1)} \wedge \dots \wedge \theta^{B_n(p_n)} = 0 \quad (2.4)$$

where $C^{A(p)}_{B_1(p_1)\dots B_n(p_n)}$ are generalized structure constants with the same symmetry as induced by permuting the θ .s in the wedge product. They can be non-zero only if:

$$p + 1 = \sum_{i=1}^n p_i \quad (2.5)$$

Equations (2.4) are self-consistent and define an FDA if and only if $dd\theta^{A(p)} = 0$ upon substitution of (2.4) into its own derivative. This procedure yields the generalized Jacobi identities of FDA.s.²

Classification of FDA and the analogue of Levi theorem: minimal versus contractible algebras A basic theorem of Lie algebra theory states that the most general Lie algebra \mathcal{A} is the semidirect product of a semisimple Lie algebra \mathcal{L} called the Levi subalgebra with $\text{Rad}(\mathcal{A})$, namely with the radical of \mathcal{A} . By definition this latter is the maximal solvable ideal of \mathcal{A} . Sullivan [2] has provided an analogous structural theorem for FDA.s. To this effect one needs the notions of *minimal FDA* and *contractible FDA*. A minimal FDA is one for which:

$$C^{A(p)}_{B(p+1)} = 0 \quad (2.6)$$

²For a review of FDA theory see [7]

This excludes the case where a $(p + 1)$ -form appears in the generalized Maurer Cartan equations as a contribution to the derivative of a p -form. In a minimal algebra all non differential terms are products of at least two elements of the algebra, so that all forms appearing in the expansion of $d\theta^{A(p)}$ have at most degree p , the degree $p + 1$ being ruled out.

On the other hand a *contractible FDA* is one where the only form appearing in the expansion of $d\theta^{A(p)}$ has degree $p + 1$, namely:

$$d\theta^{A(p)} = \theta^{A(p+1)} \quad \Rightarrow \quad d\theta^{A(p+1)} = 0 \quad (2.7)$$

A contractible algebra has a trivial structure. The basis $\{\theta^{A(p)}\}$ can be subdivided in two subsets $\{\Lambda^{A(p)}\}$ and $\{\Omega^{B(p+1)}\}$ where A spans a subset of the values taken by B , so that:

$$d\Omega^{B(p+1)} = 0 \quad (2.8)$$

for all values of B and

$$d\Lambda^{A(p)} = \Omega^{A(p+1)} \quad (2.9)$$

Denoting by \mathcal{M}^k the vector space generated by all forms of degree $p \leq k$ and C^k the vector space of forms of degree k , a minimal algebra is shortly defined by the property:

$$d\mathcal{M}^k \subset \mathcal{M}^k \wedge \mathcal{M}^k \quad (2.10)$$

while a contractible algebra is defined by the property

$$dC^k \subset C^{k+1} \quad (2.11)$$

In analogy to Levi's theorem, the first theorem by Sullivan states that: *The most general FDA is the semidirect sum of a contractible algebra with a minimal algebra*

Sullivan's first theorem and the gauging of FDA.s Twenty years ago in [6] one of us observed that the above mathematical theorem has a deep physical meaning relative to the gauging of algebras. Indeed I proposed the following identifications:

1. The *contractible generators* $\Omega^{A(p+1)} + \dots$ of any given FDA \mathbb{A} are to be physically identified with the *curvatures*
2. The Maurer Cartan equations that begin with $d\Omega^{A(p+1)}$ are *the Bianchi identities*.
3. The algebra which is gauged is the *minimal subalgebra* $\mathbb{M} \subset \mathbb{A}$.
4. The Maurer Cartan equations of the minimal subalgebra \mathbb{M} are consistently obtained by those of \mathbb{A} by setting all contractible generators to zero.

Sullivan's second structural theorem and Chevalley cohomology The second structural theorem proved by Sullivan³ deals with the structure of minimal algebras and it is constructive. Indeed it states that the most general minimal FDA \mathbb{M} necessarily contains an ordinary Lie subalgebra $\mathbb{G} \subset \mathbb{M}$ whose associated 1-form generators we can call e^I , as in equation (2.2). Additional p -form generators $A^{[p]}$ of \mathbb{M} are necessarily, according to Sullivan's theorem, in one-to-one correspondence with Chevalley $p+1$ cohomology classes $\Gamma^{[p+1]}(e)$ of $\mathbb{G} \subset \mathbb{M}$. Indeed, given such a class, which is a polynomial in the e^I generators, we can consistently write the new higher degree Maurer Cartan equation:

$$d A^{[p]} + \Gamma^{[p+1]}(e) = 0 \quad (2.12)$$

where $A^{[p]}$ is a new object that cannot be written as a polynomial in the old objects e^I . Considering now the FDA generated by the inclusion of the available $A^{[p]}$, one can inspect its Chevalley cohomology: the cochains are the polynomials in the extended set of forms $\{A, e^I\}$ and the boundary operator is defined by the enlarged set of Maurer Cartan equations. If there are new cohomology classes $\Gamma^{[p+1]}(e, A)$, then one can further extend the FDA by including new p -generators $B^{[p]}$ obeying the Maurer Cartan equation:

$$\partial B^{[p]} + \Gamma^{[p+1]}(e, A) = 0 \quad (2.13)$$

The iterative procedure can now be continued by inspecting the cohomology classes of type $\Gamma^{[p+1]}(e, A, B)$ which lead to new generators $C^{[p]}$ and so on. Sullivan's theorem states that those constructed in this way are, up to isomorphisms, the most general minimal FDA.s.

To be precise, this is not the whole story. There is actually one generalization that should be taken into account. Instead of *absolute Chevalley cohomology* one can rather consider *relative Chevalley cohomology*. This means that rather than being \mathbb{G} -singlets, the Chevalley p -cochains can be assigned to some linear representation of the Lie algebra \mathbb{G} :

$$\Omega^{\alpha[p]} = \Omega_{I_1 \dots I_p}^{\alpha} e^{I_1} \wedge \dots \wedge e^{I_p} \quad (2.14)$$

where the index α runs in some representation D :

$$D \quad : \quad T_I \rightarrow [D(T_I)]_{\beta}^{\alpha} \quad (2.15)$$

and the boundary operator is now the covariant ∇ :

$$\nabla \Omega^{\alpha[p]} \equiv \partial \Omega^{\alpha[p]} + e^I \wedge [D(T_I)]_{\beta}^{\alpha} \Omega^{\beta[p]} \quad (2.16)$$

Since $\nabla^2 = 0$, we can repeat all previously explained steps and compute cohomology groups. Each non trivial cohomology class $\Gamma^{\alpha[p+1]}(e)$ leads to new p -form generators $A^{\alpha[p]}$ which are assigned to the same \mathbb{G} -representation as $\Gamma^{\alpha[p+1]}(e)$. All successive steps go through in the same way as before and Sullivan's theorem actually states that all minimal FDA.s are obtained in this way for suitable choices of the representation D , in particular the singlet.

³For detailed explanations on this see again, apart from the original article [2] the book [7]

3 The super FDA of M theory and its cohomological structure

Sullivan's theorems have been introduced and proved for Lie algebras and their corresponding FDA extensions but they hold true, with obvious modifications, also for superalgebras \mathbb{G}_s and for their FDA extensions. Actually, in view of superstring and supergravity, it is precisely in the supersymmetric context that FDA.s have found their most relevant applications. As an illustration of the general set up let me consider the case of M-theory and of its FDA, by recalling the results of [3] and [6]. we begin by writing the complete set of curvatures, plus their Bianchi identities. This will define the complete FDA:

$$\mathbb{A} = \mathbb{M} \biguplus \mathbb{C} \quad (3.1)$$

The curvatures being the contractible generators \mathbb{C} . By setting them to zero we retrieve, according to Sullivan's first theorem, the minimal algebra \mathbb{M} . This latter, according instead to Sullivan's second theorem, has to be explained in terms of cohomology of the normal subalgebra $\mathbb{G} \subset \mathbb{M}$, spanned by the 1-forms. In this case \mathbb{G} is just the $D = 11$ superalgebra spanned by the following 1-forms:

1. the vielbein V^a
2. the spin connection ω^{ab}
3. the gravitino ψ

The higher degree generators of the minimal FDA \mathbb{M} are:

1. the bosonic 3-form $\mathbf{A}^{[3]}$
2. the bosonic 6-form $\mathbf{A}^{[6]}$.

The complete set of curvatures is given below ([3, 6]):

$$\begin{aligned} T^a &= \mathcal{D}V^a - i\frac{1}{2}\bar{\psi} \wedge \Gamma^a \psi \\ R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \\ \rho &= \mathcal{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab} \psi \\ \mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} - \frac{1}{2}\bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b \\ \mathbf{F}^{[7]} &= d\mathbf{A}^{[6]} - 15\mathbf{F}^{[4]} \wedge \mathbf{A}^{[3]} - \frac{15}{2}V^a \wedge V^b \wedge \bar{\psi} \wedge \Gamma_{ab} \psi \wedge \mathbf{A}^{[3]} \\ &\quad - i\frac{1}{2}\bar{\psi} \wedge \Gamma_{a_1\dots a_5} \psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5} \end{aligned} \quad (3.2)$$

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities, which for brevity we do not explicitly write (see [6]). The dynamical theory is defined, according to a general constructive scheme of supersymmetric theories, by the principle of rheonomy (compare with the tables 1 and 2) implemented into Bianchi identities.

3.1 Rheonomy

Indeed there is a unique rheonomic parametrization of the curvatures (3.2) which solves the Bianchi identities and it is the following one:

$$\begin{aligned}
T^a &= 0 \\
\mathbf{F}^{[4]} &= F_{a_1 \dots a_4} V^{a_1} \wedge \dots \wedge V^{a_4} \\
\mathbf{F}^{[7]} &= \frac{1}{84} F^{a_1 \dots a_4} V^{b_1} \wedge \dots \wedge V^{b_7} \epsilon_{a_1 \dots a_4 b_1 \dots b_7} \\
\rho &= \rho_{a_1 a_2} V^{a_1} \wedge V^{a_2} - i \frac{1}{2} \left(\Gamma^{a_1 a_2 a_3} \psi \wedge V^{a_4} + \frac{1}{8} \Gamma^{a_1 \dots a_4 m} \psi \wedge V^m \right) F^{a_1 \dots a_4} \\
R^{ab} &= R_{cd}^{ab} V^c \wedge V^d + i \rho_{mn} \left(\frac{1}{2} \Gamma^{abmn} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \psi \wedge V^c \\
&\quad + \bar{\psi} \wedge \Gamma^{mn} \psi F^{mnab} + \frac{1}{24} \bar{\psi} \wedge \Gamma^{abc_1 \dots c_4} \psi F^{c_1 \dots c_4}
\end{aligned} \tag{3.3}$$

The expressions (3.3) satisfy the Bianchi.s provided the space-time components of the curvatures satisfy the following constraints

$$\begin{aligned}
0 &= \mathcal{D}_m F^{mc_1 c_2 c_3} + \frac{1}{96} \epsilon^{c_1 c_2 c_3 a_1 a_8} F_{a_1 \dots a_4} F_{a_5 \dots a_8} \\
0 &= \Gamma^{abc} \rho_{bc} \\
R_{cm}^{am} &= 6 F^{ac_1 c_2 c_3} F^{bc_1 c_2 c_3} - \frac{1}{2} \delta_b^a F^{c_1 \dots c_4} F^{c_1 \dots c_4}
\end{aligned} \tag{3.4}$$

which are the space-time field equations.

4 The constrained BRST algebra from the FDA

Applying a general procedure we can obtain the explicit form of the constrained BRST algebra appropriate to any supergravity.

The first step is the construction of the standard ghost-form extension of supergravity. This has been codified in the general principle 1.1 formulated in [8] and recalled in the introduction. The correct BRST algebra is provided by replacing, in the rheonomic parametrization, of the classical supergravity curvatures each differential form with its extended ghost-form counterpart while keeping the curvature components untouched. Thus one obtains the rheonomic parametrization of the ghost-extended curvatures, whose formal definition is identical with that of the classical curvatures upon the replacements (1.3)

The above constructive principle is completely algorithmic and applies without exception to all supergravity theories. It emphasizes the fact that the BRST algebra in the positive ghost number sector is, through, a codified number os steps, deterministic consequence of the algebraic structure of the FDA, actually of the original super Lie algebra. Indeed this latter determines via cohomology its own FDA extension, then the Bianchi identities determine via rheonomy a unique parametrization of curvature in superspace and from that we obtain, also uniquely, all BRST transformations.

The next step consists of equating to zero all bosonic ghosts. This produces a unique form of a constrained BRST algebra and a unique form of constraints on the superghosts.

Once the principle has been clarified we can perform the two steps at once by considering the purely fermionic ghost-extension and then invoking principle 1.1.

Each extended curvature definition $\widehat{\mathbf{R}}_{def}^{[p]}$ and each extended curvature parametrization $\widehat{\mathbf{R}}_{par}^{[p]}$ decomposes into ghost sectors according to:

$$\begin{aligned}\widehat{\mathbf{R}}_{def}^{[p]} &= \mathbf{R}_{def}^{[p,0]} + \mathbf{R}_{def}^{[p-1,1]} + \mathbf{R}_{def}^{[p-2,2]} \\ \widehat{\mathbf{R}}_{par}^{[p]} &= \mathbf{R}_{par}^{[p,0]} + \mathbf{R}_{par}^{[p-1,1]} + \mathbf{R}_{par}^{[p-2,2]}\end{aligned}\quad (4.1)$$

where we stop at ghost number $g = 2$ since neither in the curvature definitions nor in the curvature parametrizations there appear higher than quadratic powers of the ψ forms. Then we have to impose:

$$\begin{aligned}\mathbf{R}_{def}^{[p,0]} &= \mathbf{R}_{par}^{[p,0]} \\ \mathbf{R}_{def}^{[p-1,1]} &= \mathbf{R}_{par}^{[p-1,1]} \\ \mathbf{R}_{def}^{[p-2,2]} &= \mathbf{R}_{par}^{[p-2,2]}\end{aligned}\quad (4.2)$$

The first of eq.s (4.2) is simply the rheonomic parametrization of the classical curvature we started from. The second equation defines the constrained BRST transformation of all the physical fields. The last of eq.s (4.2) defines the BRST transformation of the ghost fields (the pure spinors) when the right hand side is non zero ($\mathbf{R}_{par}^{[p-2,2]} \neq 0$) and the quadratic pure spinor constraints $\mathbf{R}_{def}^{[p-2,2]} = 0$ when the right hand side is zero $\mathbf{R}_{par}^{[p-2,2]} = 0$.

Let us write the result of these straightforward manipulations in the case of M-theory

4.1 The constrained BRST algebra of M-theory

In the case of the minimal FDA of M-theory (disregarding the 6-form) the purely fermionic ghost-extension procedure corresponds to setting:

$$\begin{aligned}V^a &\mapsto V^a \\ \mathbf{A}^{[3]} &\mapsto \mathbf{A}^{[3]} \\ \psi &\mapsto \psi + \lambda\end{aligned}\quad (4.3)$$

where λ is the commuting superghosts. Next it is convenient to introduce a Lorentz covariant formalism by splitting the ghost extended Lorentz covariant derivative in the following way:

$$\begin{aligned}\widehat{D} &= \widehat{d} + \widehat{\omega}^{ab} J_{ab} \\ &= d + s + \omega^{ab} J_{ab} + \epsilon^{ab} J_{ab} \\ &= \mathcal{D} + \mathcal{S}\end{aligned}$$

where

$$\begin{aligned}\mathcal{D} &= d + \omega^{ab} J_{ab} \quad \text{Lorentz covariant external derivative} \\ \mathcal{S} &= s + \epsilon^{ab} J_{ab} \quad \text{Lorentz covariant BRST variation}\end{aligned}\quad (4.4)$$

and where J_{ab} denotes the standard generators of the $\text{SO}(1,10)$ Lie algebra. In the above formulae ϵ^{ab} are the Lorentz-ghosts which are field dependent on the superghosts in the usual

way as the spin connection is field dependent on the gravitinos upon solving the zero torsion equation.

With these notations, from the second of equations (4.2) we obtain the BRST-transformations of the physical fields:

$$\begin{aligned}
\mathcal{S}V^a &= i\bar{\Psi}\Gamma^a\lambda \\
\mathcal{S}\Psi &= -\mathcal{D}\lambda - i\frac{1}{2}\left(\Gamma^{a_1a_2a_3}\lambda V^{a_4} + \frac{1}{8}\Gamma^{a_1\dots a_4m}\lambda V^m\right)F^{a_1\dots a_4} \\
&\equiv -\nabla\lambda \\
\mathcal{S}\mathbf{A}^{[3]} &= \bar{\Psi}\wedge\Gamma_{ab}\lambda\wedge V^a\wedge V^b
\end{aligned} \tag{4.5}$$

while from the third of eq.s (4.2) we obtain the BRST variation of the superghost and the appropriate pure spinor constraints, namely:

$$\begin{aligned}
\mathcal{S}\lambda &= 0 \\
0 &= \bar{\lambda}\Gamma^a\lambda \\
0 &= \bar{\lambda}\Gamma^{mn}\lambda V_m\wedge V_n
\end{aligned} \tag{4.6}$$

In addition from the rheonomic parametrization of the Lorentz curvatures we also learn the closure relations satisfied by the commutators and anticommutators of the operators \mathcal{D} and \mathcal{S} . They are as follows:

$$\begin{aligned}
\mathcal{S}^2 &= [\bar{\lambda}\Gamma^{mn}\lambda F^{mnab} + \frac{1}{24}\bar{\lambda}\Gamma^{abc_1\dots c_4}\lambda F^{c_1\dots c_4}]J_{ab} \\
\mathcal{D}^2 &= [R_{mn}^{ab}V^m\wedge V^n \\
&\quad + i\bar{\rho}_{mn}\left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a}\delta^{b]c} + 2\Gamma^{ab[m}\delta^{n]c}\right)\Psi\wedge V^c \\
&\quad + \bar{\Psi}\wedge\Gamma^{mn}\Psi F^{mnab} + \frac{1}{24}\bar{\Psi}\wedge\Gamma^{abc_1\dots c_4}\Psi F^{c_1\dots c_4}]J_{ab}
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\mathcal{S}\mathcal{D} + \mathcal{D}\mathcal{S} &= [i\bar{\rho}_{mn}\left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a}\delta^{b]c} + 2\Gamma^{ab[m}\delta^{n]c}\right)\lambda\wedge V^c \\
&\quad + 2\bar{\lambda}\Gamma^{mn}\Psi F^{mnab} + \frac{1}{12}\bar{\lambda}\Gamma^{abc_1\dots c_4}\Psi F^{c_1\dots c_4}]J_{ab}
\end{aligned}$$

4.2 Discussion on the constraints

We have seen that the pure constraints coming from the FDA algebra are

$$0 = \bar{\lambda}\Gamma^m\lambda, \quad 0 = \bar{\lambda}\Gamma^{mn}\lambda V_m\wedge V_n. \tag{4.8}$$

We can decompose them into irreducible representations of $\text{SO}(1,9)$ and in that case we have

$$\begin{aligned}
0 &= \bar{\lambda}\Gamma^I\lambda, & 0 &= \bar{\lambda}\Gamma^{11}\lambda, \\
0 &= \bar{\lambda}\Gamma^{IJ}\lambda V_I\wedge V_J, & 0 &= \bar{\lambda}\Gamma^{I11}\lambda V_I
\end{aligned} \tag{4.9}$$

where the indices I, J run from 0 to 9. In this way we can analyze the difference between these constraints and the pure spinor constraints in the case of type IIA superstrings [9]. In work [10] only the first set of constraints have been analyzed yielding 23 independent d.o.f. for the pure spinor field λ . However, the second type of constraints becomes necessary to ensure the BRST invariance of the M2 action. In [10] a stronger form of the constraint appeared, but in the present derivation we find only (4.8).

We would like to study the number of independent d.o.f. from (4.8). For that we use the Fierz identity for the gamma matrices in 11d and we get the relations (decomposed in the $SO(1, 9)$ representations)

$$\bar{\lambda} \Gamma^{IJ} \lambda \bar{\lambda} \Gamma_J \lambda + \bar{\lambda} \Gamma^{I11} \lambda \bar{\lambda} \Gamma_{11} \lambda = 0, \quad \bar{\lambda} \Gamma^{11J} \lambda \bar{\lambda} \Gamma_J \lambda = 0 \quad (4.10)$$

Using the first constraint $\bar{\lambda} \Gamma_J \lambda = 0$, we reduced them to the equation

$$\bar{\lambda} \Gamma^{I11} \lambda \bar{\lambda} \Gamma_{11} \lambda = 0. \quad (4.11)$$

We notice that by contracting the free index I with the 10d vielbein V_I , we do not get any condition on $\bar{\lambda} \Gamma_{11} \lambda$. However, if we assume that $\bar{\lambda} \Gamma^{11J} \lambda \neq 0$ for J orthogonal to V_J , we get that $\bar{\lambda} \Gamma_{11} \lambda = 0$ as a consequence.

Using a decomposition of the spinors $\lambda, \bar{\lambda}$ in terms of an adapted basis it can be shown that equations (4.10) implies that there are 22 independent d.o.f and this coincides with the counting of the pure spinor constraints for type IIA superstrings [9]. It is impressive that even if the structure of the constraints are different, the number of independent d.o.f. is the same. Therefore, it seems that by exploiting the complete set of constraints for the supermembrane (namely the pure spinor constraints (4.8)) one finds the agreement with the pure spinor constraints for the superstrings. It will be subject of another paper the complete discussion on the pure spinor constraints for type IIA/B in presence of RR fields from the FDA algebras [12].

5 Conclusions

In this communication the main goal has been that of illustrating the intimate relation between the pure spinor BRST algebra and the structure of the FDA underlying supergravity. This sheds new light on the quantization of superstrings á la pure spinors. This latter appears the only viable candidate to include the coupling of the string to Ramond-Ramond background fields.

References

- [1] N. Berkovits, *Super-Poincare covariant quantization of the superstring*, JHEP **0004**, 018 (2000) [arXiv:hep-th/0001035].
- [2] D. Sullivan *Infinitesimal computations in topology* Bull. de l'Institut des Hautes Etudes Scientifiques, Publ. Math. **47** (1977)

- [3] R. D'Auria and P. Fré, *Geometric supergravity in $D=11$ and its hidden supergroup* Nucl. Phys. **B201** (1982) 101.
- [4] P. A. Grassi, G. Policastro, M. Porrati and P. van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, JHEP **0210**, 054 (2002) [arXiv:hep-th/0112162]. P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, *The massless spectrum of covariant superstrings*, JHEP **0211**, 001 (2002) [arXiv:hep-th/0202123].
- [5] N. Berkovits, *Explaining pure spinor superspace*, arXiv:hep-th/0612021.
- [6] P. Fré, *Comments on the 6-index photon in $D=11$ supergravity and the gauging of free differential algebras*, Class. Quant. Grav. **1** (1984) L81.
- [7] L. Castellani, R. D'Auria, P. Fré *Supergravity and superstrings: a geometric perspective*, World Scientific, Singapore 1991.
- [8] D. Anselmi and P. Fré, *Twisted $N=2$ supergravity as topological gravity in four-dimensions*, Nucl. Phys. B **392** (1993) 401 [arXiv:hep-th/9208029].
- [9] N. Berkovits and P. S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, Nucl. Phys. B **635** (2002) 75 [arXiv:hep-th/0112160].
- [10] N. Berkovits, *Covariant quantization of the supermembrane*, JHEP **0209**, 051 (2002) [arXiv:hep-th/0201151].
- [11] P. Fré and P. A. Grassi, *Pure spinors, free differential algebras, and the supermembrane*, Nucl. Phys. B **763** (2007) 1 [arXiv:hep-th/0606171].
- [12] P. Fré and P. A. Grassi, *Pure spinor constraints in type IIA and type IIB theories from FDA and rheonomy* to appear
- [13] P. Fré and P. A. Grassi, R. D'Auria and M. Trigiante *Pure spinor string sigma model on general type II A backgrounds from FDA and rheonomy* to appear
- [14] P. Fré and P. A. Grassi, *Constrained Supermanifolds for AdS M-Theory Backgrounds*,” arXiv:0704.3413 [hep-th].